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Tunneling of a Dipolar Bose–Einstein Condensate in an Optical Lattice

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Abstract We have studied the tunneling and fluctuations of a dipolar Bose–Einstein condensate in an optical lattice, it is found that there exist the tunneling and fluctuations between lattices l and $l+1$, l and $l-1$, respectively. In particular, when the optical lattice is infinitely long and the spin excitations are in the long-wavelength limit, tunneling effects disappear between lattices l and $l+1$, and that l and $l-1$, in this case the fluctuations are a constant, and the magnetic soliton appears.

Keywords Dipolar Bose–Einstein condensate · Tunneling · Atomic number fluctuation

1 Introduction

Recent advance of experimental techniques on Bose–Einstein condensate (BEC) prompts us to closely and seriously look into theoretical possibilities which were mere imagination for theoreticians in this field. This is particularly true for spinor BEC where all hyperfine states of an atom Bose-condensed simultaneously, keeping these "spin" states degenerate and active. Recently, Barrett et al. [1] have succeeded in cooling Rb^{87} with the hyperfine state $F=1$ by all optical methods without resorting to a usual magnetic trap in which the

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internal degrees of freedom is frozen. Since the spin interaction of the Rb^{87} atomic system is ferromagnetic, based on the refined calculation of the atomic interaction parameters by Klausen et al. [2], we now obtain concrete examples of the three-component spinor BEC ($F = 1, m_F = 1, 0, -1$) for both antiferromagnetic (Na^{23}) [3] and ferromagnetic interaction cases. In the present spinor BEC the degenerate internal degrees of freedom play an essential role to determine the fundamental physical properties [4–12]. There is a rich variety of topological defect structures, which are already predicted in the earlier studies [13] on the spinor BEC. An excellent algebraic representation of the $F = 1$ BEC Hamiltonian to study the exact many-body states were constructed [14, 15], and they found that spin-exchange interactions cause a set of collective dynamic behavior of BEC. Recently, the spin wave excitation of spinor BECs, and the interaction between the spin waves have been studied [16, 17]. In this letter, we shall study the tunneling of a dipolar Bose–Einstein condensate in an optical lattice.

2 Hamiltonian of a Dipolar Bose–Einstein Condensate in an Optical Lattice

We consider a dilute gas of bosons in the optical lattice with the following Hamiltonian [17],

$$\hat{H} = \sum_n [\epsilon_n \hat{C}_n^\dagger \hat{C}_n + J_{nn+1} \hat{C}_n^\dagger \hat{C}_{n+1} + J_{nn-1} \hat{C}_n^\dagger \hat{C}_{n-1} + U_0 \hat{C}_n^\dagger \hat{C}_n^\dagger \hat{C}_n \hat{C}_n$$
$$+ U_1 (\hat{C}_{n+1}^\dagger \hat{C}_n^\dagger \hat{C}_{n+1} \hat{C}_n + \hat{C}_{n-1}^\dagger \hat{C}_n^\dagger \hat{C}_{n-1} \hat{C}_n) + U_2 \hat{C}_n^\dagger \hat{C}_n^\dagger (\hat{C}_{n+1} + \hat{C}_{n-1}) \hat{C}_n], \quad (1)$$

where $\hat{C}_n$ is the annihilation operator of a particle at the lattice site n, which is considered as being in a state described by the Wannier function $\varpi(z - z_n)$ of the lowest energy band, localized on this site; $U_0 = (4\pi\hbar^2 a)/m$ is the short-range on-site interaction given by the s-wave scattering length a. When $a > 0$ (or <0), $U_0 > 0$ (or <0) corresponds to the repulsive (or attractive) potential. U_i $(i = 1, 2)$ are the nearest-neighbor dipole-dipole interactions. $\epsilon_n = \epsilon_0 + \epsilon_{ext}$ is the total energy of each lattice site, and

$$\epsilon_0 = \int \varpi^*(\mathbf{r} - \mathbf{r_n})[-(\hbar^2/2m)\nabla^2 + V_{\text{opt}}]\varpi(\mathbf{r} - \mathbf{r_n})dz, \quad (2)$$

$$\epsilon_{\text{ext}} = \int \varpi^*(\mathbf{r} - \mathbf{r_n}) V_{\text{ext}} \varpi(\mathbf{r} - \mathbf{r_n})dz, \quad (3)$$

$$J_{nn\pm1} = \int \varpi^*(\mathbf{r} - \mathbf{r_n})[-(\hbar^2/2m)\nabla^2 + V_{\text{opt}} + V_{\text{ext}}]\varpi(\mathbf{r} - \mathbf{r_{nn\pm1}})dz, \quad (4)$$

here ϵ_{ext} describes an energy offset of each lattice site, $J_{nn\pm1}$ is the hopping term which describes the nearest-neighbor tunneling; $V_{\text{opt}} = V_0 \sin^2(2\pi z/\lambda)$ is the optical lattice potential, λ is the light wave length, V_{ext} is an external potential such as the gravity or magnetic traps.

3 Tunneling of a Dipolar Bose–Einstein Condensate in an Optical Lattice

In this section, we shall study the tunneling of a dipolar Bose–Einstein condensate in an optical lattice. The nonlinear interactions in Hamiltonian (1) generate the nonlinear magnetic excitations such as mixing of spin waves or magnetic solitons depending on the initial setup of the state of the spin system. Here we are interested in the study of tunneling induced by the nonlinear interactions. The ideal case is that the spin system in the optical lattice should initially be prepared in a spin-coherent state $|\psi\rangle = |\{\psi_l\}\rangle = \Pi_l|\psi_l\rangle$, with $|\psi_l\rangle = \exp(-|\psi_l|^2/2)\exp(-\psi_l C^\dagger)|0\rangle$. The vacuum state $|0\rangle$ is the ground state of the BEC in the optical lattice, i.e., $|0\rangle = |GS\rangle = |N, N\rangle$.

Under the spin coherent state and using the time-dependent variation principle, the nonlinear motion equation of atomic number $\xi_l = \langle\psi|C_l^\dagger C_l|\psi\rangle$ on the lattice l can be derived as

$$\begin{aligned} i\hbar\frac{\partial\xi_l}{\partial t} &= J_{nn+1}(\xi_{l+1}\xi_l^* - \xi_l\xi_{l-1}^*) + J_{nn-1}(\xi_{l-1}\xi_l^* - \xi_l\xi_{l+1}^*) \\ &\quad + U_2(|\xi_l|^2\xi_l^*\xi_{l+1} - |\xi_{l-1}|^2\xi_l\xi_{l-1}^* + |\xi_l|^2\xi_l^*\xi_{l-1} - |\xi_{l+1}|^2\xi_l\xi_{l+1}^*). \end{aligned} \tag{5}$$

Similarly, we can obtain the motion equation of atomic number $\mu_{l+1} = \langle\psi|C_{l+1}^\dagger C_{l+1}|\psi\rangle$ and $\nu_{l-1} = \langle\psi|C_{l-1}^\dagger C_{l-1}|\psi\rangle$ on the lattice $l+1$ and $l-1$, respectively:

$$\begin{aligned} i\hbar\frac{\partial\mu_{l+1}}{\partial t} &= J_{nn+1}(\mu_{l+2}\mu_{l+1}^* - \mu_{l+1}\mu_l^*) + J_{nn-1}(\mu_l\mu_{l+1}^* - \mu_{l+1}\mu_{l+2}^*) \\ &\quad + U_2(|\mu_{l+1}|^2\mu_{l+2}\mu_{l+1}^* - |\mu_l|^2\mu_{l+1}\mu_l^* + |\mu_{l+1}|^2\mu_l\mu_{l+1}^* \\ &\quad - |\mu_{l+2}|^2\mu_{l+1}\mu_{l+2}^*), \end{aligned} \tag{6}$$

and

$$\begin{aligned} i\hbar\frac{\partial\nu_{l-1}}{\partial t} &= J_{nn+1}(\nu_l\nu_{l-1}^* - \nu_{l-1}\nu_{l-2}^*) + J_{nn-1}(\nu_{l-2}\nu_{l-1}^* - \nu_{l-1}\nu_l^*) \\ &\quad + U_2(|\nu_{l-1}|^2\nu_l\nu_{l-1}^* - |\nu_{l-2}|^2\nu_{l-1}\nu_{l-2}^* + |\nu_{l-1}|^2\nu_{l-2}\nu_{l-1}^* - |\nu_l|^2\nu_{l-1}\nu_l^*), \end{aligned} \tag{7}$$

According to (5–7), we see that $\frac{\partial}{\partial t}(\xi_l - \mu_{l+1}) \neq \frac{\partial}{\partial t}(\xi_l - \nu_{l-1})$, which means that the tunnelings between lattices l and $l+1$, l and $l-1$ are general different. In particular, when the optical lattice is infinitely long and the spin excitations are in the long-wavelength limit, one has $\xi_l = \xi_{l+1} = \xi_{l-1}$, $\mu_l = \mu_{l+1} = \mu_{l-1}$, and $\nu_l = \nu_{l+1} = \nu_{l-1}$ in the continuum limit approximation, this shows that there does not exist the tunneling effect between lattices l and $l+1$, and that l and $l-1$. Correspondingly, the magnetic soliton appears.

In order to give an explicit physical meaning for the above results, we now discuss the atomic number fluctuation in lattice site l, $l+1$, and $l-1$, respectively. The fluctuation of atomic number on lattice l can be written as

$$\begin{aligned} (\Delta N_l)^2 &= \langle\psi|(C_l^\dagger C_l)^2|\psi\rangle - (\langle\psi|C_l^\dagger C_l|\psi\rangle)^2 \\ &= \langle\psi|C_l^{\dagger 2}C_l^2|\psi\rangle + \langle\psi|C_l^\dagger C_l|\psi\rangle - (\langle\psi|C_l^\dagger C_l|\psi\rangle)^2. \end{aligned} \tag{8}$$

Similarly, the fluctuations of atomic number on the lattices $l+1$ and $l-1$ are

$$(\Delta N_{l+1})^2 = \langle\psi|C_{l+1}^{\dagger 2}C_{l+1}^2|\psi\rangle + \langle\psi|C_{l+1}^\dagger C_{l+1}|\psi\rangle - (\langle\psi|C_{l+1}^\dagger C_{l+1}|\psi\rangle)^2, \tag{9}$$

$$(\Delta N_{l-1})^2 = \langle\psi|C_{l-1}^{\dagger 2}C_{l-1}^2|\psi\rangle + \langle\psi|C_{l-1}^\dagger C_{l-1}|\psi\rangle - (\langle\psi|c_{l-1}^\dagger C_{l-1}|\psi\rangle)^2. \tag{10}$$

The motion equations of the $\xi_l = \langle\psi|C_l^\dagger C_l|\psi\rangle$, $\mu_{l+1} = \langle\psi|C_{l+1}^\dagger C_{l+1}|\psi\rangle$ and $\nu_{l-1} = \langle\psi|C_{l-1}^\dagger C_{l-1}|\psi\rangle$ have been given above (see (5–7)), we here give the motion equations of $\zeta_l = \langle\psi|C_l^{\dagger 2}C_l^2|\psi\rangle$, $\eta_{l+1} = \langle\psi|C_{l+1}^{\dagger 2}C_{l+1}^2|\psi\rangle$ and $\varsigma_{l-1} = \langle\psi|C_{l-1}^{\dagger 2}C_{l-1}^2|\psi\rangle$ as follows:

$$\begin{aligned} i\hbar\frac{\partial\zeta_l}{\partial t} &= 2J_{nn+1}(|\zeta_l|^2\zeta_{l+1}\zeta_l^* - |\zeta_l|^2\zeta_l\zeta_{l-1}^*) + 2J_{nn-1}(|\zeta_l|^2\zeta_{l-1}\zeta_l^* - |\zeta_l|^2\zeta_l\zeta_{l+1}^*) \\ &\quad + 2U_2(|\zeta_l|^4\zeta_{l+1}\zeta_l^* + |\zeta_l|^2\zeta_{l+1}\zeta_l^* - |\zeta_l|^2|\zeta_{l-1}|^2\zeta_l\zeta_{l-1}^* \\ &\quad + |\zeta_l|^4\zeta_{l-1}\zeta_l^* + |\zeta_l|^2\zeta_{l-1}\zeta_l^* - |\zeta_l|^2|\zeta_{l+1}|^2\zeta_l\zeta_{l+1}^*), \end{aligned} \tag{11}$$

$$\begin{aligned} i\hbar\frac{\partial\eta_{l+1}}{\partial t} &= 2J_{nn+1}(|\eta_{l+1}|^2\eta_{l+2}\eta_{l+1}^* - |\eta_{l+1}|^2\eta_{l+1}\eta_l^*) \\ &\quad + 2J_{nn-1}(|\eta_{l+1}|^2\eta_l\eta_{l+1}^* - |\eta_{l+1}|^2\eta_{l+1}\eta_{l+2}^*) \\ &\quad + 2U_2(|\eta_{l+1}|^4\eta_{l+2}\eta_{l+1}^* + |\eta_{l+1}|^2\eta_{l+2}\eta_{l+1}^* - |\eta_l|^2|\eta_{l+1}|^2\eta_{l+1}\eta_l^* \\ &\quad + |\eta_{l+1}|^4\eta_l\eta_{l+1}^* + |\eta_{l+1}|^2\eta_l\eta_{l+1}^* - |\eta_{l+2}|^2|\eta_{l+1}|^2\eta_{l+1}\eta_{l+2}^*), \end{aligned} \tag{12}$$

and

$$\begin{aligned} i\hbar\frac{\partial\varsigma_{l-1}}{\partial t} &= 2J_{nn+1}(|\varsigma_{l-1}|^2\varsigma_l\varsigma_{l-1}^* - |\varsigma_{l-1}|^2\varsigma_{l-1}\varsigma_{l-2}^*) \\ &\quad + 2J_{nn-1}(|\varsigma_{l-1}|^2\varsigma_{l-2}\varsigma_{l-1}^* - |\varsigma_{l-1}|^2\varsigma_{l-1}\varsigma_l^*) \\ &\quad + 2U_2(|\varsigma_{l-1}|^4\varsigma_l\varsigma_{l-1}^* + |\varsigma_{l-1}|^2\varsigma_l\varsigma_{l-1}^* - |\varsigma_{l-1}|^2|\varsigma_{l-2}|^2\varsigma_{l-1}\varsigma_{l-2}^* \\ &\quad + |\varsigma_{l-1}|^4\varsigma_{l-2}\varsigma_{l-1}^* + |\varsigma_{l-1}|^2\varsigma_{l-2}\varsigma_{l-1}^* - |\varsigma_{l-1}|^2\varsigma_{l-1}\varsigma_l^{*3}). \end{aligned} \tag{13}$$

Combining (5–7) and (8–13), we see that the motion equations of the fluctuations of the atomic number are different, which are the reasons of appearing the tunneling effects between lattices l and $l+1$, and that of l and $l-1$. Especially, when the optical lattice is infinitely long and the spin excitations are in the long-wavelength limit, the atomic number fluctuation discussed above is a constant, respectively, and there is not the tunneling effect.

4 Conclusions

In summary, we have studied the tunneling and fluctuations of a dipolar Bose–Einstein condensate in an optical lattice. It is found that there exist the tunneling effects and fluctuations between lattices l and $l+1$, l and $l-1$, respectively. In particular, when the optical lattice is infinitely long and the spin excitations are in the long-wavelength limit, tunneling effects disappear between lattices l and $l+1$, and that l and $l-1$, in this case the fluctuations are a constant, and the magnetic soliton appears.

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